Dark energy and quantum vacuum fluctuations

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Abstract

It is suggested that the vacuum expectation of the quantum vacuum energy-momentum is zero, but quantum fluctuations give rise to a space-time curvature equivalent to that of a cosmological constant or dark energy. Calculations within quantized gravity, following a few plausible hypotheses, provide results compatible with cosmological observations.

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The observed accelerated expansion of the universe[1] is assumed to be due to a positive mass density and negative pressure, constant throughout space and time, which is popularly known as "dark energy". The mass (or energy) density, ρ_{DE}, and the presure, p_{DE}, are [2]

$$\rho_{DE} \simeq -p_{DE} \simeq 10^{-26} kg/m. \tag{1}$$

(Throughout this paper I shall use units $c = \hbar = 1$, but write explicitly Newton´s constant, G, for the sake of clarity.)

The current wisdom is to identify the dark energy with the cosmological constant introduced by Einstein in 1917 or, what is equivalent in practice, to assume that it corresponds to the quantum vacuum. Indeed the equality $\rho_{DE} = -p_{DE}$ is appropriate for the vacuum (in Minkowski space, or when the space-time curvature is small) because it is invariant under Lorentz transformations. A problem appears however when one attempts to estimate the value of ρ_{DE}. In fact if the dark energy is really due to the quantum vacuum

it seems difficult to understand why the mass density is not either strictly zero or of the order of Planck´s density, that is

$$\rho_{DE} \sim \frac{c^5}{G^2\hbar} \simeq 10^{97} kg/m, \tag{2}$$

which is about 123 orders larger than the observed eq.(1).

In this paper I explore the possibility that the quantum vacuum energy is indeed zero but the quantum fluctuations give rise to a curvature of space-time similar to the one produced by a constant classical (non-fluctuating) mass density and pressure as given by eq.(1) . More correctly stated, the hypothesis is that the quantum vacuum consists a set of interacting relativistic quantum fields giving rise to an energy-momentum quantum tensor operator, $\widehat{T}_{\mu}^{\nu}\left(x^{\eta}\right)$, whose vacuum expectation is zero at any space-time point. That is

$$\left\langle 0 \left| \widehat{T}_{\mu}^{\nu}\left(x^1, x^2, x^3, x^4\right) \right| 0 \right\rangle = 0, \tag{3}$$

where $\mid 0\rangle$ is the state-vector of the vacuum and x^1, x^2, x^3, x^4 the coordinates of a space-time point, which I shall label collectively x^{η} in the following. In contrast the existence of quantum fluctuations implies that the vacuum expectation of the product of the components at two space-time points may not be zero, that is

$$\left\langle 0 \left| \widehat{T}_{\mu}^{\nu}\left(x^{\eta}\right) \widehat{T}_{\sigma}^{\lambda}\left(x^{\zeta}\right) \right| 0 \right\rangle \neq 0. \tag{4}$$

The effect of the quantum vacuum on the curvature of spacetime should be calculated within the framework of quantized gravity. This means assuming that the vacuum is characterized by a metric tensor operator $\widehat{g}_{\mu\nu}\left(x^{\eta}\right)$ which is related to the energy-momentum tensor operator $\widehat{T}_{\mu}^{\nu}\left(x^{\eta}\right)$ by some equations to be specified. An obvious constraint on these equations is that they will agree with Einstein´s equations in the classical limit.

My aim is to see whether the quantum vacuum fluctuations, plus the matter content of the universe, may give rise to the same spce-time curvature as in the standard model. The most common description of space-time in cosmology involves the use of the Robertson-Walker-Friedman metric. However for our purposes it is more convenient to use a local frame with “curvature” coordinates appropriate for spherical symmetry around some arbitrary point of space, that is

$$ds^2 = A\left(r,t\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 - B\left(r,t\right) dt^2. \tag{5}$$

For a relatively small region around the origin of the coordinate system the expressions of A and B are simple, namely[3]

$$
\begin{aligned}
A &= 1+\frac{8\pi}{3}Gr^{2}\left(\rho_{mat}+\rho_{DE}\right)+O\left(r^{4}\right), \\
B &= 1+\frac{8\pi}{3}Gr^{2}\left(\frac{1}{2}\rho_{mat}-\rho_{DE}\right)+O\left(r^{4}\right),
\end{aligned} \tag{6}
$$

where ρ_{mat} is the density of cold matter, either baryonic or dark, ρ_{DE} is the density of dark energy as given in eq.(1) and I ignore the (small) contributions of radiation and cold matter pressure. Thus the task is to reproduce eqs.(6) without a real dark energy density, but including the effect of quantum vacuum fluctuations.

Although a complete quantum gravity theory, not yet available, would be needed for a rigorous treatment, we may derive some relevant results via introducing a few plausible hypotheses. For the sake of clarity I will write explicitly these assumptions as "propositions".

Proposition 1 *The global properties of space-time, e.g. the accelerated expansion of the universe or the mean curvature of space if any, may be obtained from the vacuum expectation value of the metric tensor operator, that expectation being treated as if it was an actual classical metric tensor.*

That is I will assume that the following (classical, c-number) metric tensor

$$
g_{\mu\nu}\left(x^{\eta}\right)=\left\langle 0\left|\widehat{g}_{\mu\nu}\left(x^{\eta}\right)\right|0\right\rangle, \tag{7}
$$

determines the global properties of space-time. Obviously the quantum fluctuations of the metric cannot be derived from $g_{\mu\nu}$. In particular

$$
\left\langle 0\left|\widehat{g}_{\mu\nu}\left(x^{\eta}\right)\widehat{g}_{\lambda\sigma}\left(x^{\zeta}\right)\right|0\right\rangle \neq g_{\mu\nu}\left(x^{\eta}\right)g_{\lambda\sigma}\left(x^{\zeta}\right).
$$

In order to make a comparison with eqs.(5) and (6), I will write eqs.(7) in curvature coordinates. Thus they are specified as follows.

Proposition 2 *The vacuum expectation values of the diagonal metric coefficients are*

$$
\begin{aligned}
\left\langle 0\left|\widehat{g}_{11}\left(x^{\eta}\right)\right|0\right\rangle &= A(r,t),\ \left\langle 0\left|\widehat{g}_{22}\left(x^{\eta}\right)\right|0\right\rangle = r^{2}, \\
\left\langle 0\left|\widehat{g}_{33}\left(x^{\eta}\right)\right|0\right\rangle &= r^{2}\sin^{2}\theta,\ \left\langle 0\left|\widehat{g}_{44}\left(x^{\eta}\right)\right|0\right\rangle = B(r,t),
\end{aligned} \tag{8}
$$

the vacuum expectations of non-diagonal coefficients being zero.

These relations are a consequence of our choice of coordinates plus the assumption that the distribution of matter is isotropic on the large scale, a standard approximation in cosmology.

Now I will state the energy-momentum content of the universe as follows.

Proposition 3 *In addition to the contribution of the quantum vacuum, there is a cold matter density,* ρ_{mat}*, (either baryonic or dark) which is uniform in space but depends on time. Thus the total density and pressure operators are*

$$\widehat{\rho}\left(\mathbf{x},t\right)=\widehat{I}\rho_{mat}\left(t\right)+\widehat{\rho}_{vac}\left(\mathbf{x},t\right),\ \widehat{p}\left(\mathbf{x},t\right)=\widehat{p}_{vac}\left(\mathbf{x},t\right), \tag{9}$$

where $\widehat{I}$ *is the identity operator and* $\mathbf{x}$ *is the vector with polar coordinates* $\left(r,\theta,\phi\right)$.

For the sake of simplicity I neglect the small contributions of the hot matter (radiation) and the pressure associated to cold matter.

The next task will be to relate the coefficients A and B with the distribution of matter plus the quantum vacuum fluctuations. As a guide I shall start from relations valid in classical gravity for a space-time of spherical symmetry in curvature coordinates. These relations are[4]

$$\begin{aligned} A\left(r,t\right) &= \left(1-\frac{2Gm\left(r\right)}{r}\right)^{-1},\ m\left(r\right)\equiv 4\pi\int_0^r\rho\left(x\right)x^2dx, \\ B(r,t) &= \exp\left[2G\int_0^r\frac{m\left(x\right)+4\pi x^3p\left(x\right)}{x^2-2Gm\left(x\right)x}dx\right], \end{aligned} \tag{10}$$

ρ being the density and p the pressure of a perfect fluid.

For the purposes of this paper it is enough to work to second order in Newton´s constant, G. Thus I will write the first eq.(10) in the form

$$A\left(r,t\right)=1+\frac{2Gm}{r}+\frac{4G^2m^2}{r^2}+O\left(G^3\right). \tag{11}$$

Similarly the second eq.(10) may be written

$$\begin{aligned} B(r,t) &= 1+2G\int_0^r\left[x^{-2}m(x)+4\pi xp\left(x\right)\right]dx \\ &\quad +4G^2\int_0^r\left[x^{-3}m(x)^2+4\pi m(x)p(x)\right]dx \\ &\quad +2G^2\left[\int_0^r\left[x^{-2}m(x)+4\pi xp\left(x\right)\right]dx\right]^2+O\left(G^3\right). \end{aligned} \tag{12}$$

In order to pass to quantized gravity we should write A and B as vacuum expectations of expressions involving the operators $\widehat{\rho}\left(\mathbf{x},t\right)$ and $\widehat{p}\left(\mathbf{x},t\right)$ introduced in eqs.(9) . In the absence of any clear hint, I might suppose that those expressions are similar to eqs.(10) and (12) . Thus I will assume the following.

Proposition 4 *The vacuum expectations of the metric coefficients are related to the quantum operators of density, $\widehat{\rho}$, and pressure, $\widehat{p}$, (the diagonal elements of the energy-momentum tensor operator $\widehat{T}_{\mu}^{\nu}\left(x^{\eta}\right)$) by*

$$\begin{aligned} A\left(r,t\right) &\simeq 1+\left\langle 0\left|\frac{2G}{r}\widehat{m}(r)+\frac{4G^{2}}{r^{2}}\widehat{m}(r)^{2}\right|0\right\rangle , \\ \widehat{m}(r) &\equiv \int_{\left|\mathbf{x}\right|\leq r}\widehat{\rho}\left(\mathbf{x}\right)d^{3}\mathbf{x} \end{aligned} \tag{13}$$

$$\begin{aligned} B(r,t) &\simeq 1+\frac{2G}{r}\left\langle 0\left|\int_{0}^{r}x^{-2}\widehat{m}(x)dx+\int_{\left|\mathbf{x}\right|\leq r}x^{-1}\widehat{p}\left(\mathbf{x}\right)d^{3}\mathbf{x}\right|0\right\rangle +B_{2}(r,t) \\ B_{2}(r,t) &= 2G^{2}\left\langle 0\left|2\int_{0}^{r}x^{-3}\widehat{m}(x)^{2}dx+\int_{\left|\mathbf{x}\right|\leq r}x^{-2}\left[\widehat{m}(x)\widehat{p}\left(\mathbf{x}\right)+\widehat{p}\left(\mathbf{x}\right)\widehat{m}(x)\right]d^{3}\mathbf{x}\right|0\right\rangle \\ &\quad +2G^{2}\left\langle 0\left|\left[\int_{0}^{r}x^{-2}\widehat{m}(x)dx+\int_{\left|\mathbf{x}\right|\leq r}x^{-1}\widehat{p}\left(\mathbf{x}\right)d^{3}\mathbf{x}\right]^{2}\right|0\right\rangle , \end{aligned} \tag{14}$$

Here the operators $\widehat{\rho}$ and $\widehat{p}$ may depend on time, but this dependence is not explicitly exhibited. It may be realized that the quantum operators $\widehat{\rho}$ and $\widehat{p}$ appear always in symmetrical ordering.

The next hypothesis refers to the correlations between the quantum fluctuations at two different points but equal times (we do not need them at different times).

Proposition 5 *The correlations between the vacuum operators of density and pressure at two space points and equal times depend only on the distance between the points, that is they might be written*

$$\begin{aligned} \left\langle 0\left|\widehat{\rho}_{vac}\left(\mathbf{x}\right)\widehat{\rho}_{vac}\left(\mathbf{y}\right)\right|0\right\rangle &= f_{\rho\rho}\left(\left|\mathbf{x}-\mathbf{y}\right|\right), \ \left\langle 0\left|\widehat{p}_{vac}\left(\mathbf{x}\right)\widehat{p}_{vac}\left(\mathbf{y}\right)\right|0\right\rangle =f_{pp}\left(\left|\mathbf{x}-\mathbf{y}\right|\right), \\ f_{\rho p}\left(\left|\mathbf{x}-\mathbf{y}\right|\right) &= \frac{1}{2}\left\langle 0\left|\widehat{\rho}_{vac}\left(\mathbf{x}\right)\widehat{p_{vac}}\left(\mathbf{y}\right)+\widehat{p}_{vac}\left(\mathbf{y}\right)\widehat{\rho}_{vac}\left(\mathbf{x}\right)\right|0\right\rangle . \end{aligned} \tag{15}$$

This hypothesis is consistent with the homogeneity and isotropy at large scale (cosmological principle). Actually the distance between points should involve the metric, but I may assume that the approximation of Minkowski metric is good enough in this case (in fact $|A(\mathbf{x},t)-1| << 1, |B(\mathbf{x},t)-1| << 1$ for any $\mathbf{x}$ and t.)

If I insert eqs.(9) into eq.(13) I get, taking eqs.(3) and (15) into account,

$$\begin{aligned} A(r,t) &\simeq 1+\frac{8\pi}{3}G\rho_{mat}(t)\,r^2+\frac{64\pi^2}{9}G^2\rho_{mat}(t)^2\,r^4 \\ &+4G^2r^{-2}\int_{|\mathbf{x}|\leq r}d^3\mathbf{x}\int_{|\mathbf{y}|\leq r}d^3\mathbf{y}\;\, f_{\rho\rho}(|\mathbf{x}-\mathbf{y}|)\,, \end{aligned} \tag{16}$$

As expected from eqs.(3) the leading contribution from the vacuum fluctuations is of order G^2. It is possible to perform the angular integrals if I define the new function $L_{\rho\rho}(x,y)$ by

$$L_{\rho\rho}(x,y)\equiv xy\int_0^{\pi}f_{\rho\rho}(|\mathbf{x}-\mathbf{y}|)\sin\theta d\theta=\int_{|x-y|}^{x+y}f_{\rho\rho}(z)\,zdz, \tag{17}$$

where

$$z\equiv|\mathbf{x}-\mathbf{y}|=\sqrt{x^2+y^2-2xy\cos\theta},$$

Thus I get for the last term of eq.(16)

$$A_{vac}=8\pi^2G^2I_1,\;\, I_1\equiv 4r^{-2}\int_{\mathbf{0}}^{r}xdx\int_{\mathbf{0}}^{r}ydy\;L_{\rho\rho}(x,y). \tag{18}$$

Similarly I define

$$L_{\rho\rho}(x,y)\equiv xy\int_0^{\pi}f_{\rho\rho}(|\mathbf{x}-\mathbf{y}|)\sin\theta d\theta,\;\, L_{\rho\rho}(x,y)\equiv xy\int_0^{\pi}f_{\rho\rho}(|\mathbf{x}-\mathbf{y}|)\sin\theta d\theta. \tag{19}$$

Thus inserting eqs.(9) into eq.(14) I obtain, taking eqs.(3) and (15) into account,

$$B(r,t)\simeq 1+\frac{4\pi}{3}G\rho_{mat}(t)\,r^2+\frac{8\pi^2}{3}G^2\rho_{mat}(t)^2\,r^4+B_{vac}, \tag{20}$$

where

$$B_{vac}=8\pi^2G^2\sum_{k=2}^{6}I_k, \tag{21}$$

and the integrals I_k are

$$
\begin{aligned}
I_2 &= 4\int_0^r x^{-3}dx \int_0^x udu \int_0^x vdv\; L_{\rho\rho}(u,v), \\
I_3 &= 4\int_0^r x^{-1}dx \int_0^x udu\; L_{\rho p}(x,u), \\
I_4 &= 2\int_0^r x^{-2}dx \int_0^r y^{-2}dy \int_0^x udu \int_0^y vdv\; L_{\rho\rho}(u,v), \\
I_5 &= 2\int_0^r x^{-2}dx \int_0^x udu \int_0^r dv\; L_{\rho p}(u,v), \\
I_6 &= 2\int_0^r dx \int_0^r dy\; L_{pp}(x,y). \qquad (22)
\end{aligned}
$$

We cannot proceed further until we fix the functions $L(x,y)$ or, what is equivalent, the functions $f\left(|\mathbf{x}-\mathbf{y}|\right)$.This is made with our next hypothesis, which is justified as follows. By comparison of the quantity A_{vac}, eq.(18), with eq.(6) we see that agreement requires that the integral I_1 should be proportional to r^2. Similar scaling is needed for the integrals I_2 to I_6. As a consequence the functions $f_{\rho\rho}\left(|\mathbf{x}-\mathbf{y}|\right), f_{\rho p}\left(|\mathbf{x}-\mathbf{y}|\right)$ and $f_{pp}\left(|\mathbf{x}-\mathbf{y}|\right)$, should scale with distance as r^{-2}. This leads to the following assumption.

Proposition 6 *The correlations between components of the vacuum energy-momentum tensor at two different point and equal times should be proportional to the inverse of the square of the distance between the points. That is*

$$
\begin{aligned}
f_{\rho\rho}\left(|\mathbf{x}-\mathbf{y}|\right) &= C_{\rho\rho}\left|\mathbf{x}-\mathbf{y}\right|^{-2},\; f_{\rho p}\left(|\mathbf{x}-\mathbf{y}|\right) = C_{\rho p}\left|\mathbf{x}-\mathbf{y}\right|^{-2}, \\
f_{pp}\left(|\mathbf{x}-\mathbf{y}|\right) &= C_{pp}\left|\mathbf{x}-\mathbf{y}\right|^{-2}, \qquad (23)
\end{aligned}
$$

where $C_{\rho\rho}, C_{\rho p}$ and C_p are constant quantities, that is independent of $\mathbf{x}$ and t.

After that the calculation of the integrals I_1 to I_6 is straightforward although lengthy. I get

$$
\begin{aligned}
I_1 &= 2C_{\rho\rho}r^2,\; I_2 = C_{\rho\rho}r^2,\; I_3 = 2C_{\rho p}r^2,\; I_4 = \left(\frac{4}{3}\log 2 - \frac{1}{3}\right)C_{\rho\rho}r^2, \\
I_5 &= \left(\frac{5}{3}\log 2 - \frac{4}{3}\right)C_{\rho p}r^2,\; I_6 = \log 2 C_{pp}r^2. \qquad (24)
\end{aligned}
$$

Agreement of A_{vac}, eq.(18) , and B_{vac}, eq,(21) , with eq.(6) will be obtained if

$$16\pi^2 G^2 C_{\rho\rho} = \frac{8\pi}{3} G\rho_{DE} \Rightarrow \rho_{DE} = 6\pi G C_{\rho\rho}. \tag{25}$$

$$3\pi G\left[\frac{2}{3}\left(2\log 2+1\right)C_{\rho\rho}+\frac{1}{3}\left(5\log 2+4\right)C_{\rho p}+2\log 2 C_{pp}\right]=-\rho_{DE}. \tag{26}$$

This is only possible if the correlations of the quantum vacuum fluctuations, given by eqs,(15) and (23) , fulfil

$$\frac{4}{3}\left(\log 2+2\right)C_{\rho\rho}+\frac{1}{3}\left(5\log 2+4\right)C_{\rho p}+2\log 2 C_{pp}=0. \tag{27}$$

In order to fix the quantities $C_{\rho\rho}, C_{\rho p}$ and C_p I need an additional assumption, which I will make as follows.

Proposition 7 *The density correlation and the pressure correlation are equal.*

Thus I obtain a rather simple relation between the said quantities, namely

$$C_{\rho\rho} = C_{pp} = -\frac{1}{2} C_{\rho p}.$$

In summary the calculation *suggests* that the “dark” energy (or mass) density, ρ_{DE}, and pressure, p_{DE}, are fictitious but the curvature of space-time is real and it is the same that would be produced by a mass density and a pressure as in eq.(1) . The value of the mass density, ρ_{DE}, may be obtained as a product of Newton´s constant, G, times some factor, Q, which depends on the properties of the vacuum quantum fields, likely those of the standard model of elementary particles. We might estimate the order of the parameter Q by means of a dimensionally correct combination of the Planck constant, $\hbar$, the speed of light, c, and a typical mass of elementary particles, m. Consequently, in order that ρ_{DE} has dimensions of energy density, we shall assume

$$\rho_{DE} \sim G\frac{m^6 c^2}{\hbar^4}. \tag{28}$$

Unfortunately eq.(28) is very sensitive to the actual value of the unknown mass m. Likely the mass lies somewhere between the electron and the proton mass, which gives

$$10^{-35} kg/m^3 \lesssim \rho_{DE} \lesssim 10^{-24} kg/m^3,$$

a rather wide interval. In any case the results of our calculations are compatible with the observed value, eq.(1) , and far from the value eq.(2) . Consequently our results are consistent with the assumption that dark energy is just a fictitious energy and pressure appropriate in order to parametrize the curvature of space-time due to quantum vacuum fluctuations.

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